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Complete description of a quantum system at a given time

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Abstract. A generalization of the description of a quantum system in the time interval between two measurements is presented. A new concept of a *generalized state* is introduced. Generalized states yield a complete description of a quantum system when information about the system is available both from the past and from the future. The formalism of generalized states provides a natural language for describing many peculiar situations. In particular, situations in which one can ascertain the result of a measurement of any one of several non-commuting variables are analysed. 'Weak' measurements on quantum systems described by generalized states are discussed. The relation between 'weak' and 'strong' measurements is investigated.

1. Introduction

In the standard quantum theory the most complete description of a quantum system at a given time is given by its state vector or, when the system is correlated to some other systems, by its density matrix. This is the maximal information about the system based on the results of the experiments performed in the past. In this work we introduce a formalism for a complete description of a quantum system at a given time based on the results of experiments performed both before and after this time. Our description provides the maximal possible information about the quantum system. The formalism presented here is a generalization of the approach pioneered by Aharonov, Bergman and Lebowitz [1] and employed by us for developing the concept of *weak measurements* [2]. We have described a quantum system between two measurements by two state vectors: the usual one, evolving from the time of the latest complete measurement in the past, and the other one evolving backward in time from the time of the earliest complete measurement in the future. We generalize, here, this description to the interesting situation in which these two state vectors, the one from the past and the other from the future, are correlated. We introduce a new concept—*the generalized state*—a superposition of the pairs of the above state vectors. We discuss quantum systems described by generalized states in peculiar situations such as a spin-$\frac{1}{2}$ particle which can be found with probability one in the state $\sigma_{\hat{\xi}} = 1$ for several directions $\hat{\xi}$, or a *single* particle which can be found with probability one in *several* separate boxes, etc. The concept of *weak values* of quantum variables, which was defined for the systems described by the pair of states, is extended here to the systems described by generalized states. This makes possible the analysis of the connection between 'weak' and 'strong' (conventional) measurements.

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To avoid possible misinterpretations we want to stress from the outset that our work neither contradicts ordinary quantum theory nor extends it in the sense of new physical laws. It is possible, in principle, to obtain all the results presented in this work using the standard formalism. However, using only quantum states evolving towards the future, it is more difficult to prove these results, and in some cases it is almost impossible to foresee them.

The plan of this paper is as follows. In section 2 we derive the formula for the probability of results of a measurement performed between two other measurements. In section 3 we develop the concept of a generalized state. In section 4 we analyse quantum systems described by generalized states for which, at a certain time, several non-commuting variables have definite values. In particular we obtain the maximal set of directions for which we can ascertain the spin components of a spin-$\frac{1}{2}$ particle. Section 5 is devoted to the concept of 'weak values' [2] of quantum variables for systems described by generalized states. In section 6 we prove some theorems about the connection between 'weak' and 'strong' (conventional) values of quantum variables. Section 7 concludes the paper with some remarks regarding the meaning and the significance of the results which have been derived.

2. The probability of results of a measurement performed between two other measurements

At time t between two measurements (performed at times t_1 and t_2, $t_1 < t < t_2$) we propose to describe a quantum system using two state vectors. Let us assume that the measurement at time t_1 created a non-degenerate eigenstate $|\psi_1(t_1)\rangle$. This state has the standard time evolution so that at time t:

$$|\psi_1(t)\rangle = U(t_1, t)|\psi_1(t_1)\rangle \tag{1}$$

where U is the time evolution operator. Similarly, the measurement at time t_2, whose outcome corresponds to the non-degenerate eigenstate $|\psi_2(t_2)\rangle$, yields at the time t a backward evolving state vector $\langle\psi_2(t_2)|$ (we denote it by 'bra', instead of 'ket'):

$$\langle\psi_2(t)| = \langle\psi_2(t_2)|U(t, t_2). \tag{2}$$

According to our proposal, the system at any time t, $t_1 < t < t_2$, is described by the pair of state vectors (1) and (2). We shall call this pair of state vectors *the generalized state*, and write it as

$$\Psi(t) \equiv \langle\psi_2(t)||\psi_1(t)\rangle. \tag{3}$$

We shall discuss the concept of the generalized state further in the next section. Now we address the problem of finding the probability of a result of a quantum measurement performed between two other measurements. This was first discussed by Aharonov, Leibowitz and Bergman in 1964 [1]. They proved that in the above situation the measurement at time t of a non-degenerate operator C yields the eigenvalue c_n with the probability

$$\text{prob}(C = c_n) = \frac{|\langle\psi_2(t)|C = c_n\rangle|^2 |\langle C = c_n|\psi_1(t)\rangle|^2}{\Sigma_i |\langle\psi_2(t)|C = c_i\rangle|^2 |\langle C = c_i|\psi_1(t)\rangle|^2}. \tag{4}$$

We shall now generalize this formula for an operator C which has degenerate eigenstates. We shall show that for this case the probability for the outcome c_n is given

by

$$\text{prob}(C = c_n) = \frac{|\langle\psi_2(t)|\boldsymbol{P}_{C=c_n}|\psi_1(t)\rangle|^2}{\Sigma_i\,|\langle\psi_2(t)|\boldsymbol{P}_{C=c_i}|\psi_1(t)\rangle|^2} \tag{5}$$

where $\boldsymbol{P}_{C=c_i}$ is the projection operator on the space of eigenstates with eigenvalue c_i:

$$\boldsymbol{P}_{C=c_i} = \sum_{\alpha} |\phi_{i,\alpha}\rangle\langle\phi_{i,\alpha}|. \tag{6}$$

Here $\{|\phi_{i,\alpha}\rangle\}$ is a complete set of eigenstates with eigenvalue c_i.

The desired probability of finding $C = c_n$ is a conditional probability. Using the Bayes theorem we can express it as a function of conditional probabilities for events depending only on earlier events. These last conditional probabilities can be calculated using only the forward evolving state vectors. The left-hand side of equation (5), $\text{prob}(C = c_n)$, is:

$$\text{prob}[C(t) = c_n|\psi_{in} = \psi_1(t_1),\ \psi_f = \psi_2(t_2)]$$
$$= \frac{\text{prob}[C(t) = c_n|\psi_{in} = \psi_1(t_1)]\ \text{prob}[\psi_f = \psi_2(t_2)|C(t) = c_n,\ \psi_{in} = \psi_1(t_1)]}{\Sigma_i\ \text{prob}[C(t) = c_i|\psi_{in} = \psi_1(t_1)]\ \text{prob}[\psi_f = \psi_2(t_2)|C(t) = c_i,\ \psi_{in} = \psi_1(t_1)]}. \tag{7}$$

Two types of probabilities appear on the right-hand side of equation (7). The first is given by

$$\text{prob}[C(t) = c_i|\psi_{in} = \psi_1(t_1)] = \|\boldsymbol{P}_{C=c_i}U(t_1, t)|\psi_1(t_1)\rangle\|^2 = \sum_{\alpha} |\langle\phi_{i,\alpha}|\psi_1(t)\rangle|^2 \tag{8}$$

where we used definitions (1) and (6). The second conditional probability is $\text{prob}[\psi_f = \psi_2(t_2)|C(t) = c_i,\ \psi_{in} = \psi_1(t_1)]$. We note that when the outcome of the measurement at time t is c_i, the state vector of the system changes from $|\psi_1(t)\rangle$ to $|\psi_1'\rangle$, which is given by

$$|\psi_1'\rangle = \frac{\Sigma_\alpha\,|\phi_{i,\alpha}\rangle\langle\phi_{i,\alpha}|\psi_1(t)\rangle}{\Sigma_\alpha\,|\langle\phi_{i,\alpha}|\psi_1(t)\rangle|^2}. \tag{9}$$

Thus, the second conditional probability is

$$\text{prob}[\psi_f = \psi_2(t_2)|C = c_i,\ \psi_{in} = \psi_1(t_1)] = |\langle\psi_f = \psi_2(t_2)|\,U(t, t_2)|\psi_1'\rangle|^2$$
$$= \frac{|\langle\psi_2(t)|\boldsymbol{P}_{C=c_i}|\psi_1(t)\rangle|^2}{\Sigma_\alpha\,|\langle\phi_{i,\alpha}|\psi_1(t)\rangle|^2}. \tag{10}$$

Substituting equations (8) and (10) into equation (7) yields, finally, equation (5).

Note that equation (5) yields the probability of the result $C = c_n$ for the *preselected and postselected* ensemble, i.e. only systems which were prepared at time t_1 in the state $|\psi_1(t_1)\rangle$ and were found at time t_2 in the state $|\psi_2(t_2)\rangle$ are taken into account.

The above discussion shows that the state vectors $|\psi_1(t)\rangle$ and $\langle\psi_2(t)|$ are sufficient to calculate the probability for any outcome of all possible measurements at an *intermediate time* t. The state vector evolving toward the past $\langle\psi_2(t)|$ does not exist in the standard approach. If, following the standard approach, we limit ourselves only to forward evolving state vectors then, in order to find $\text{prob}[C(t) = c_n|\psi_{in} = \psi_1(t_1),\ \psi_f = \psi_2(t_2)]$, we have to perform a more complicated calculation. Specifically, for each outcome c_i we have to calculate $\text{prob}[\psi_f = \psi_2(t_2)|C = c_i]$. The calculation includes the time evolution of all possible states $|\phi_{i,\alpha}\rangle$ from time t to time t_2. This is to be contrasted

with our approach where we have to calculate the evolution in the time period (t, t_2) of only one (backward evolving) state vector $\langle\psi_2(t)|$. Therefore, introducing $\langle\psi_2(t)|$ through the definition (2) is certainly preferable for this type of problem.

As an example, we shall analyse a curious situation in which *one* particle can be found with certainty in N (!) separate boxes [3]. Let us consider $N+1$ boxes. We denote the state of the particle when it is in the box number i by $|i\rangle$. At time t_1 the particle is measured to be in the state

$$|\psi_1(t_1)\rangle = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} |i\rangle. \tag{11}$$

At time t_2 the particle is measured to be in the state

$$|\psi_2(t_2)\rangle = \frac{1}{\sqrt{N^2-N+1}} \left(\sum_{i=1}^{N} |i\rangle - (N-1)|N+1\rangle \right). \tag{12}$$

If in the time interval $[t_1, t_2]$ the Hamiltonian is zero, then for any intermediate time t $(t_1 < t < t_2)$ the generalized state (3) is:

$$\Psi(t) = \frac{1}{\sqrt{N^2-N+1}} \left(\sum_{i=1}^{N} \langle i| - (N-1)\langle N+1| \right) \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} |i\rangle. \tag{13}$$

By opening the box i we perform a measurement with two possible outcomes; we denote them by 'in i' when the particle is found in the box, and 'out of i' when the box is found to be empty. These outcomes correspond to the following projection operators:

$$\boldsymbol{P}_{(\text{in } i)} \equiv |i\rangle\langle i| \tag{14}$$

$$\boldsymbol{P}_{(\text{out of } i)} \equiv \sum_{k \neq i} |k\rangle\langle k|. \tag{15}$$

The generalized state (13), as we shall now show, describes a situation for which if we open any *one* of the first N boxes we certainly find the particle there. Indeed, the formula (5) for the probability to find the particle in the box i, $i \neq N+1$ yields

$$\text{prob(in } i) = \frac{|\langle\psi_2(t)|\boldsymbol{P}_{(\text{in } i)}|\psi_1(t)\rangle|^2}{|\langle\psi_2(t)|\boldsymbol{P}_{(\text{in } i)}|\psi_1(t)\rangle|^2 + |\langle\psi_2(t)|\boldsymbol{P}_{(\text{out of } i)}|\psi_1(t)\rangle|^2} = 1. \tag{16}$$

(The second term in the denominator $|\langle\psi_2(t)|\boldsymbol{P}_{(\text{out of } i)}|\psi_1(t)\rangle|^2$ vanishes when we substitute in it equations (13) and (15).)

So, in spite of the fact that we have only one particle in the above situation, we find this particle with probability one in any one of the first N boxes. If we find the particle in one of the boxes, we shall not find it later (after time t, but before t_2) in any other. Indeed, the generalized state for the time period (t, t_2) becomes

$$\Psi = \frac{1}{\sqrt{N^2-N+1}} \left(\sum_{k=1}^{N} \langle k| - (N-1)\langle N+1| \right) |i\rangle \tag{17}$$

where i is the number of the box which was opened first. Equation (5) shows that for generalized state (17) the probability to find the particle in a box j, $j \neq i$, vanishes.

No contradiction arises if we open all boxes simultaneously. This corresponds to a measurement of a non-degenerate operator whose eigenvalues correspond to $N+1$ projection operators $\boldsymbol{P}_{(\text{in } i)}$, $i = 1, 2, \ldots, N+1$, given by equation (14). Now equation (5) (or (4)) yields, for generalized state (13), the probability $1/[(N-1)^4+N]$ for finding the particle in any of the boxes from 1 to N and the probability $(N-1)^4/[(N-1)^4+N]$ for finding the particle in the box $N+1$.

3. Generalized states

We named the pair of state vectors in equation (3) a 'generalized state'. However, equation (3) corresponds only to a particular ('product') type of generalized states. The true justification for introducing this new concept comes from the possibility of superposing such pairs of states, i.e. creating the generalized state of the ('superposition') type:

$$\Psi \equiv \sum_{i=1}^{N} \alpha_i \langle \psi_i || \phi_i \rangle. \tag{18}$$

Situations in which the system at time t cannot be described by generalized state (3) arise when at times t_1 and t_2 ($t_1 < t < t_2$) certain measurements were performed on a composite system which includes our system as well as some external system. Then, the composite system is described by the generalized state of the 'product' type (3), but our system by itself is not. If during the time interval (t_1, t_2) there is no interaction with the external part of the composite system, then at any time t, $t_1 < t < t_2$, our system can be described by the superposition state (18).

We shall next describe the *operational definition* of such generalized states. To this end we need to perform measurements on a composite system which includes our system as well as an external system. The external system has a set of orthogonal states $|i\rangle_{ex}$ and we assume for simplicity that the Hamiltonian for the external system is zero†. Specifically, consider the situation in which at time t_1 we have prepared the state

$$|\Psi_1(t_1)\rangle \equiv \sum_{i=1}^{N} a_i |i\rangle_{ex} |\phi_i\rangle \tag{19}$$

and at time t_2 we have found the state

$$|\Psi_2(t_2)\rangle \equiv \sum_{i=1}^{N} b_i |i\rangle_{ex} |\psi_i\rangle. \tag{20}$$

The generalized state $\langle \Psi_2(t) || \Psi_1(t) \rangle$ of the composite system then corresponds to the generalized state (18) for our system with coefficients α_i given in terms of a_i and b_i by

$$\mathcal{N} a_i b_i^* = \alpha_i \tag{21}$$

where $\mathcal{N}$ is a normalization factor. Its value does not affect the results presented in this paper. We shall use the normalization condition

$$\sum_{i,j=1}^{N} \alpha_i \alpha_j^* \langle \phi_j | \phi_i \rangle \langle \psi_i | \psi_j \rangle = 1. \tag{22}$$

Generalized states are closely connected with the *two-time states*, which have been introduced by Aharonov and Albert [4]. Two-time measurements which prepare two-time states also create generalized states. For the two-time states it is fruitful to define a scalar product [5]. The same definition is applicable for generalized states and the normalization condition (22) corresponds to the natural requirement that the scalar product of a generalized state with itself is equal to one.

† If the Hamiltonian of the external system is not zero, the only modification is that at time t_2 in the state (13) we have to use a different set of orthogonal states: $|i(t_2)\rangle \equiv U(t_1, t_2)|i(t_1)\rangle$.

Equation (5) gives the probability for the results of a measurement performed between two other measurements. It applies to any measurement on a system which is described by the special 'product'-type generalized state (3). Our next task is to find the corresponding formula for quantum systems described by the generalized state (18). A natural generalization of equation (5) is

$$\text{prob}(C=c_n)=\frac{|\sum_{i=1}^{N}\alpha_i\langle\psi_i|\boldsymbol{P}_{C=c_n}|\phi_i\rangle|^2}{\sum_k|\sum_{i=1}^{N}\alpha_i\langle\psi_i|\boldsymbol{P}_{C=c_k}|\phi_i\rangle|^2}. \tag{23}$$

We proceed to show that equation (23) is indeed correct. The generalized state (18) arises from a product type state (3), $\boldsymbol{\Psi}=\langle\Psi_1||\Psi_2\rangle$, when $\langle\Psi_1|$ and $|\Psi_2\rangle$ refer to the composite system described above. For example,

$$\boldsymbol{\Psi}=\langle\Psi_2||\Psi_1\rangle \qquad |\Psi_1\rangle=\sum_{i=1}^{N}\alpha_i|i\rangle_{\text{ex}}|\phi_i\rangle \qquad \langle\Psi_2|=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\langle\psi_i|\langle i|_{\text{ex}}. \tag{24}$$

Since the operator C acts only on the variables of the system itself, it corresponds to a degenerate operator of the composite system: $C=\{C\}_{\psi}\times\{1\}_{\text{ex}}$. Recalling that the Hamiltonian for the external system is assumed to vanish in the time interval (t_1, t_2), we can substitute equation (24) in equation (5):

$$\begin{aligned}\text{prob}(C=c_n)&=\frac{|1/\sqrt{N}\sum_{i=1}^{N}\langle i|_{\text{ex}}\langle\psi_i|\boldsymbol{P}_{C=c_n}\sum_{i=1}^{N}\alpha_i|i\rangle_{\text{ex}}|\phi_i\rangle|^2}{\sum_k|1/\sqrt{N}\sum_{i=1}^{N}\langle i|_{\text{ex}}\langle\psi_i|\boldsymbol{P}_{C=c_k}\sum_{i=1}^{N}\alpha_i|i\rangle_{\text{ex}}|\phi_i\rangle|^2}\\&=\frac{|\sum_{i=1}^{N}\alpha_i\langle\psi_i|\boldsymbol{P}_{C=c_n}|\phi_i\rangle|^2}{\sum_k|\sum_{i=1}^{N}\alpha_i\langle\psi_i|\boldsymbol{P}_{C=c_k}|\phi_i\rangle|^2}.\end{aligned} \tag{25}$$

Thus, we have completed the derivation of equation (23).

In the definition of the generalized state (18) $\{\langle\psi_i|\}$ and $\{|\phi_i\rangle\}$ could be arbitrary sets of states. Equation (23) is correct even when $\{\langle\psi_i|\}$ and $\{|\phi_i\rangle\}$ are not orthonormal bases. if we take the *same* orthogonal bases for both state vectors, $\{|\xi_i\rangle\}$ and $\{\langle\xi_i|\}$, then the generalized state (18) takes the form

$$\boldsymbol{\Psi}=\sum_{i,j=1}^{N}\beta_{ij}\langle\xi_i||\xi_j\rangle. \tag{26}$$

In this case the formula (23) for the probability of the result $C=c_n$ will be of the form

$$\text{prob}(C=c_n)=\frac{|\sum_{i,j=1}^{N}\beta_{ij}\langle\xi_i|\boldsymbol{P}_{C=c_n}|\xi_j\rangle|^2}{\sum_k|\sum_{i,j=1}^{N}\beta_{ij}\langle\xi_i|\boldsymbol{P}_{C=c_k}|\xi_j\rangle|^2}. \tag{27}$$

We can find the orthogonal bases $\{|\psi_i'\rangle\}$ and $\{\langle\phi_i'|\}$ in which $\boldsymbol{\Psi}$ takes the *canonical* form [6]

$$\boldsymbol{\Psi}=\sum_{i=1}^{N}\alpha_i'\langle\psi_i'||\phi_i'\rangle. \tag{28}$$

The set of absolute values of the coefficients $\{|\alpha_i'|\}$ is defined uniquely by the generalized state (18). In this basis the normalization condition (22) obtains the simpler form

$$\sum_{i=1}^{N}|\alpha_i'|^2=1. \tag{29}$$

The normalization condition (22) also has a simple form when the generalized state is represented by the matrix β of equation (26). It reads as

$$\text{tr}\,\beta\beta^{\dagger}=1. \tag{30}$$

4. Quantum systems described by a generalized state for which several non-commuting variables have definite values

In the standard approach to quantum theory, operators with no common eigenstates cannot have definite values simultaneously. For example, there is no state vector of a spin-$\frac{1}{2}$ particle for which we can predict with certainty the result of measuring σ_x and the result of measuring σ_y when one of these measurements is performed. In our approach, where we are allowed to have information both from the future and from the past, this is no longer the case. There are situations in which we can ascertain the result of measurement of any one of two non-commuting operators. This is simply achieved as follows: before the time in question we measure one operator and after this time, the other. Thus, measuring at time t_1 σ_x and at time t_2 σ_y, we can ascertain the outcome of the spin component measurement performed at any intermediate time t, $t_1 < t < t_2$, provided the Hamiltonian is zero: if σ_x were measured, the result has to be the same as the result of our first measurement, and if σ_y were measured, the result has to be the same as the result of the second measurement.

There are situations in which we can ascertain the results of the measurements of even more than two non-commuting operators. For example, we have shown how to ascertain the values of σ_x, σ_y and σ_z of a spin-$\frac{1}{2}$ particle [7]. In fact, there are situations in which we can ascertain the results of measurements of an infinite number of different operators. We next describe a situation in which the result of measuring the spin component of a spin-$\frac{1}{2}$ particle is 'up' with probability one for a continuum of directions. For this to be the case the spin-$\frac{1}{2}$ particle has to be described by a specific generalized state as we proceed to show.

To this end we first consider the generalized state for which the result of a spin component measurement is 'up' for at least three different directions. The generalized state of a spin-$\frac{1}{2}$ particle is given by

$$\Phi = a_{\uparrow\uparrow}\langle\uparrow_z|\uparrow_z\rangle + a_{\uparrow\downarrow}\langle\uparrow_z|\downarrow_z\rangle + a_{\downarrow\uparrow}\langle\downarrow_z|\uparrow_z\rangle + a_{\downarrow\downarrow}\langle\downarrow_z|\downarrow_z\rangle \tag{31}$$

where we used the basis in which σ_z is diagonal. The unitary transformation of the usual, forward-evolving state vector $|\rangle$ under a rotation parametrized by the Euler angles α, β, γ, is given in the above 'z' representation by the following matrix:

$$U = \begin{Bmatrix} \cos\beta/2\, e^{i(\alpha+\gamma)/2} & \sin\beta/2\, e^{-i(\alpha-\gamma)/2} \\ -\sin\beta/2\, e^{i(\alpha-\gamma)/2} & \cos\beta/2\, e^{-i(\alpha+\gamma)/2} \end{Bmatrix}. \tag{32}$$

The generalized state for which the measurement of σ_z yields the result $\sigma_z = 1$ with probability one, is given by Φ of equation (31) with $a_{\downarrow\downarrow} = 0$, $a_{\uparrow\uparrow} \neq 0$. Let us assume that Φ is such a generalized state. Then, in the basis which is rotated by the angles α and β, we obtain

$$a'_{\downarrow\downarrow} = a_{\uparrow\uparrow}\sin^2\frac{\beta}{2} - \sin\frac{\beta}{2}\cos\frac{\beta}{2}(a_{\uparrow\downarrow}\, e^{-i\alpha} + a_{\downarrow\uparrow}\, e^{i\alpha}). \tag{33}$$

We are searching for directions $\hat{z}'$ for which the outcome of the spin component measurement is 'up' with probability one, i.e. $a'_{\downarrow\downarrow} = 0$. We can choose the overall phase in such a way that $a_{\uparrow\uparrow}$ is real. Then, the requirement $a'_{\downarrow\downarrow} = 0$ leads to

$$(a_{\uparrow\downarrow} - a^*_{\downarrow\uparrow})\, e^{-i\alpha} = e^{i\alpha}(a^*_{\downarrow\downarrow} - a_{\downarrow\uparrow}). \tag{34}$$

The direction $\hat{z}'$ is defined by α and β; therefore, in order to obtain more than two distinct directions, the requirement $a'_{\downarrow\downarrow} = 0$ should not fix α and β. Hence, we must

have $a_{\uparrow\downarrow} = a^*_{\downarrow\uparrow}$. We can show that in this case there is a direction $\hat{\eta}$ such that in the η basis the generalized state Φ is

$$\Phi = b_{\uparrow\uparrow}\langle\uparrow_\eta||\uparrow_\eta\rangle - b_{\downarrow\downarrow}\langle\downarrow_\eta||\downarrow_\eta\rangle \qquad b_{\uparrow\uparrow}, b_{\downarrow\downarrow} > 0. \tag{35}$$

The direction $\hat{\eta}$ for which equation (35) holds is obtained by rotating the z axis by Euler angles α, β such that $a_{\uparrow\downarrow}\,e^{-i\alpha}$ is real and $\tan\beta = 2a_{\uparrow\downarrow}\,e^{-i\alpha}/a_{\uparrow\uparrow}$.

Now, for any direction $\hat{\xi}$ at angle $\beta' = 2\tan^{-1}\sqrt{b_{\downarrow\downarrow}/b_{\uparrow\uparrow}}$ with respect to the direction $\hat{\eta}$, the measurement of the spin yields $\sigma_\xi = 1$ with probability one. Indeed, the calculation of the coefficients of the decomposition of Φ in the ξ basis gives $a'_{\downarrow\downarrow} = 0$. The additional requirement $a'_{\uparrow\uparrow} \neq 0$ yields $\beta' \neq 90°$. Therefore, our continua correspond to cones whose axis is the direction $\hat{\eta}$ and whose opening angles are

$$\theta = 4\tan^{-1}\sqrt{\frac{b_{\downarrow\downarrow}}{b_{\uparrow\uparrow}}}. \tag{36}$$

The normalization of the generalized state (29), reduces the number of independent parameters to one: parameter χ replaces $b_{\uparrow\uparrow}$ and $b_{\downarrow\downarrow}$,

$$\Phi = \cos\chi\,\langle\uparrow_\eta||\uparrow_\eta\rangle - \sin\chi\,\langle\downarrow_\eta||\downarrow_\eta\rangle \qquad \chi \in \left(0, \frac{\pi}{2}\right). \tag{37}$$

Thus, we showed that if there are three different directions for which we can ascertain that the result of the measurement of the spin is 'up' with probability one, then necessarily there is a continuum of such directions and the geometrical form of this continuum is a cone. The form of the corresponding generalized state is given by equation (37) where the direction $\hat{\eta}$ lies along the internal axis of the cone. The opening angle of the cone in the new notation is $\theta = 4\tan^{-1}\sqrt{\tan\chi}$.

Now we shall apply this geometrical picture for discussing the following problem [8]: 'What is the maximal set of directions for which we can ascertain the value of the spin component using measurements before and after the time in question?' More specifically, which measurements do we have to perform in order to be able to construct a table of statements such as: 'if the spin were measured in the direction $\hat{\xi}_1$ then, with probability one, the result is 'up' (or 'down'); if the spin were measured in the direction $\hat{\xi}_2$ then, with probability one, the result is . . . , and so on'? Since we cannot ensure any specific result of our second verification measurement performed on the composite system, we have to be able to ascertain the values of the above spin components for all possible outcomes. We showed that if, for a certain outcome, we can ascertain the value of spin components for at least three different directions, then we can do it for a continuum of directions forming a cone. Thus, to achieve our goal, any outcome of the second measurement should correspond to a cone. The directions along which we can always ascertain the value of the spin projection have to be common to all these cones. More precisely, we consider the lines which are common to all the 'double' cones: the cones together with the continuation of their rays beyond the apex†.

In the special case discussed in the previous footnote in which we were able to ascertain the values of σ_x, σ_y and σ_z, there were four possible outcomes for the second measurement which was performed on a composite system of two spin-$\frac{1}{2}$ particles.

† If the spin is 'up' with probability one in a certain direction it is 'down' with certainty in the opposite one. The cone of directions with the result 'up' together with the cone of directions with the result 'down' make the 'double' cone. One double cone corresponds to two generalized states depending on which half of the double cone has the result 'up' for the spin component measurement.

These outcomes corresponded to four double cones having tetrahedral symmetry: the angles between all pairs of axes of the cones were equal. The opening angles of the cones were also the same: $\theta = 4\tan^{-1}\sqrt{\tan\pi/12}$. The cones intersect exactly in three mutually orthogonal lines which can be chosen as the x, y and z axes.

Returning to the more general case when the above-mentioned table is to be constructed we note that the complete set of outcomes of the second measurement in the procedure for creation of the generalized state may correspond just to three, geometrically different, double cones. While four distinct double cones may have at most three common lines, three double cones may have at most four distinct common lines. From the geometrical picture of three intersecting double cones it is clear that these four lines are not arbitrary. Three lines are arbitrary but the fourth line is fixed by the first three: it must have the same angles with respect to two out of the three lines as the remaining line has. This is equivalent to the requirement which was found by Ben-Menahem [8]:

$$\sum_{i=1}^{4} \boldsymbol{n}_i = \boldsymbol{0} \tag{38}$$

where $\boldsymbol{n}_i$ are the unit vectors along our four lines. Ben-Menahem derived this result working in the framework of standard quantum mechanics. Our approach using generalized states explains it in a simpler and more direct way. Indeed, it has been this approach that led us to raise this kind of problem when we first realized that the values of more than two non-commuting observables at a given time can be ascertained using measurements before and after the time in question.

5. Weak values of quantum variables for systems described by generalized states

Recently we introduced [9] and discussed [2] the concept of weak values of a quantum system at the time interval between two measurements. For a system, prepared in the state $|\psi_1\rangle$ and then found in the state $|\psi_2\rangle$, the weak value of a variable A in an intermediate time was defined as

$$A_w \equiv \frac{\langle\psi_2|A|\psi_1\rangle}{\langle\psi_2|\psi_1\rangle}. \tag{39}$$

Now we can generalize the definition of weak values for systems described by the generalized state of the form (18), $\sum_{i=1}^{N} \alpha_i\langle\psi_i||\phi_i\rangle$. The above definition, equation (39), corresponds to the particular case of a generalized state of the product type given by equation (3). We recall that any generalized state which cannot be represented in the form of a product (3) for the system itself can still be represented as such product type state but for a composite system which includes also some external system. Considering the procedure which prepares the generalized state (18) and which uses the verification measurements of states described by equation (24): we obtain

$$A_w = \frac{1/\sqrt{N}\sum_{i=1}^{N}\langle i|_{ex}\langle\psi_i|A\sum_{i=1}^{N}\alpha_i|i\rangle_{ex}|\phi_i\rangle}{1/\sqrt{N}\sum_{i=1}^{N}\langle i|_{ex}\langle\psi_i|\sum_{i=1}^{N}\alpha_i|i\rangle_{ex}|\phi_i\rangle} = \frac{\sum_{i=1}^{N}\alpha_i\langle\psi_i|A|\phi_i\rangle}{\sum_{i=1}^{N}\alpha_i\langle\psi_i|\phi_i\rangle}. \tag{40}$$

Equation (40) is then the appropriate generalization of the definition of the weak value (39) for an arbitrary generalized state.

The weak value A_w is the effective value of A for any sufficiently weak coupling between our system and an external system which depends on A. The condition of weakness of the interaction is, roughly, that the generalized state does not change significantly at the period of time between the preparation and the verification measurements. To see this, consider the von Neumann [10] measurement-type coupling given by the Hamiltonian

$$H = -g(t)qA \tag{41}$$

where $g(t)$ is a normalized function with compact time support and q is some canonical variable. The Hamiltonian (41) represents a 'weak' coupling if the initial state of the external system is, for example, a Gaussian $e^{-q^2/4\Delta^2}$ with a spread which fulfils the following requirement [2]:

$$(2\Delta)^n \frac{\Gamma(n/2)}{(n-2)!} |(A^n)_w - (A_w)^n| \ll 1 \qquad \text{for all } n \geq 2. \tag{42}$$

Indeed, since the generalized state of the form (18) can be associated with the generalized state of the form (3) for the larger system, we can repeat, in order to derive equation (42), the argument of [2] where the product-type generalized state was considered.

If the generalized state is of the form (26), $\Psi = \sum_{i,j=1}^{N} \beta_{ij} \langle \xi_i || \xi_j \rangle$, where identical bases were utilized for both state vectors evolving forward and backward in time, then the weak value has the following form:

$$A_w = \frac{\sum_{i,j=1}^{N} \beta_{ij} \langle \xi_i | A | \xi_j \rangle}{\sum_{i,j=1}^{N} \beta_{ij} \langle \xi_i | \xi_j \rangle} = \frac{\mathrm{tr}(\beta A)}{\mathrm{tr}\,\beta}. \tag{43}$$

In general, there is no restriction (except for normalization) on the matrix β_{ij} which represents the generalized state. However, when the weak value is considered, we must have $\mathrm{tr}\,\beta \neq 0$. Indeed, $\mathrm{tr}\,\beta$ represents the scalar product between the states of the composite system which are verified by the first and the second measurement creating the generalized state. Between these measurements the only coupling was that of the weak measurement. This measurement is characterized by the requirement that the state vector does not change significantly. Thus, the probability of the appropriate outcome of the second measurement is given, approximately, by the square of the scalar product, i.e. by $|\mathrm{tr}\,\beta|^2$. Therefore, in this situation the probability to create the generalized state with $\mathrm{tr}\,\beta = 0$ is essentially 0. We can state it in a simpler, but more formal, way: if $\mathrm{tr}\,\beta = 0$ then the weakness condition (42) cannot be fulfilled and, so, there is no reason to consider the weak value.

6. The relation between 'weak' and 'strong' values of a quantum variable for a system described by a generalized state

In this section we shall prove that if the outcome of a strong (i.e. precise) measurement is known with certainty, then it is equal to the outcome of a corresponding weak measurement. We shall show that for a class of dichotomic variables the inverse theorem is also true. Finally, we shall show how we can use these theorems to clarify the results presented in section 4.

Let us consider the generalized state (18), $\Sigma_{i=1}^{N} \alpha_i \langle \psi_i | | \phi_i \rangle$, such that the probability of finding the result $A = a_n$ in a strong measurement of A is one. Then equation (25) becomes

$$\text{prob}(A = a_n) = \frac{|\Sigma_{i=1}^{N} \alpha_i \langle \psi_i | \boldsymbol{P}_{A=a_n} | \phi_i \rangle|^2}{\Sigma_k |\Sigma_{i=1}^{N} \alpha_i \langle \psi_i | \boldsymbol{P}_{A=a_k} | \phi_i \rangle|^2} = 1. \tag{44}$$

From this it follows that

$$\sum_{i=1}^{N} \alpha_i \langle \psi_i | \boldsymbol{P}_{A=a_k} | \phi_i \rangle = 0 \qquad \text{for all } k \neq n. \tag{45}$$

Now let us consider for this case the weak value of A. We can calculate it by substituting the decompositions of the operator A and the identity operator,

$$A = \sum_k a_k \boldsymbol{P}_{A=a_k} \tag{46a}$$

$$I = \sum_k \boldsymbol{P}_{A=a_k} \tag{46b}$$

into equation (40). Thus, taking into account equation (45), we obtain

$$A_w = \frac{\Sigma_{i=1}^{N} \alpha_i \langle \psi_i | A | \phi_i \rangle}{\Sigma_{i=1}^{N} \alpha_i \langle \psi_i | \phi_i \rangle} = \frac{\Sigma_{i=1}^{N} \alpha_i \langle \psi_i | \Sigma_k a_k P_{A=a_k} | \phi_i \rangle}{\Sigma_{i=1}^{N} \alpha_i \langle \psi_i | \Sigma_k P_{A=a_k} | \phi_i \rangle} = a_n. \tag{47}$$

The above result means that *whenever a strong measurement yields a given outcome with probability one, the weak measurement yields the same outcome.*

For the class of dichotomic variables (such as projection operators, spin component of a spin-$\frac{1}{2}$ particle, etc.) we can prove the inverse theorem: *if the weak value of a dichotomic variable equals to one of its eigenvalues, then the outcome of a strong measurement of this variable is equal to that eigenvalue with probability one.* Indeed, if a variable A has just two distinct eigenvalues a_1 and a_2, and if for the generalized state (18), $A_w = a_1$, then equation (40) becomes

$$\frac{\Sigma_{i=1}^{N} \alpha_i \langle \psi_i | a_1 \boldsymbol{P}_{A=a_1} | \phi_i \rangle + \Sigma_{i=1}^{N} \alpha_i \langle \psi_i | a_2 \boldsymbol{P}_{A=a_2} | \phi_i \rangle}{\Sigma_{i=1}^{N} \alpha_i \langle \psi_i | \boldsymbol{P}_{A=a_1} | \phi_i \rangle + \Sigma_{i=1}^{N} \alpha_i \langle \psi_i | \boldsymbol{P}_{A=a_2} | \phi_i \rangle} = a_1. \tag{48}$$

When the weak value is defined (i.e. the denominator in equation (48) does not vanish), we can, by simple algebraic manipulations of equation (48), find that

$$\sum_{i=1}^{N} \alpha_i \langle \psi_i | \boldsymbol{P}_{A=a_2} | \phi_i \rangle = 0. \tag{49}$$

This equation shows that the probability to find the value a_2 vanishes, and therefore a strong measurement of A necessarily yields $A = a_1$.

We shall derive now the results of section 4, pertaining to the situation for which we can ascertain the value of the spin component of a spin-$\frac{1}{2}$ particle for more than two directions, in yet another way.

The 'linearity' property of weak values, $C_w = A_w + B_w$ for $C \equiv A + B$, allows a geometrical picture for the generalized states. The geometry is in the vector space of the complete set of quantum variables (operators) for our system. By the complete set we mean that every operator can be expressed in the unique way as a linear combination of operators from the set. Thus, the weak values of the operators from the complete set yield the weak values of all operators.

For a spin-$\frac{1}{2}$ particle the operators σ_x, σ_y and σ_z are such a set and we have a geometry in the familiar three-dimensional space. Each generalized state corresponds

to a vector in this three-dimensional space with components equal to the weak values of σ_x, σ_y and σ_z. We shall call this vector a 'weak vector'. The weak value of a spin component in an arbitrary direction, then, is given by the projection of the weak vector on this direction. Recall that the weak value, in general, is a complex number. It is convenient to decompose the weak vector into the sum of two vectors: the 'real' and the 'imaginary' vectors:

$$\boldsymbol{\sigma}_w = \boldsymbol{\sigma}_{1w} + i\boldsymbol{\sigma}_{2w} \tag{50}$$

where $\boldsymbol{\sigma}_{1w}$ and $\boldsymbol{\sigma}_{2w}$ are real. Recall also that there are no restrictions on the weak values of the spin component, thus $\boldsymbol{\sigma}_{1w}$ and $\boldsymbol{\sigma}_{2w}$ might have any direction and any value†.

The spin component of a spin-$\frac{1}{2}$ particle is a dichotomic variable with two eigenvalues of ± 1. According to the theorems proved above, the weak value of a spin component in a certain direction is equal to 1 (-1) if, and only if, the strong measurement yields this value with probability one. Thus, the direction for which we can ascertain the value of the spin component has to be such that it is perpendicular to $\boldsymbol{\sigma}_{2w}$, and such that the projection of $\boldsymbol{\sigma}_{1w}$ on this direction is equal to ± 1.

This geometrical picture helps us to classify all possible situations. If $|\boldsymbol{\sigma}_{1w}| < 1$, then there is no direction for which we can ascertain the value of the spin component. If $|\boldsymbol{\sigma}_{1w}| = 1$ and $\boldsymbol{\sigma}_{2w}$ is perpendicular to $\boldsymbol{\sigma}_{1w}$, then there is only one direction (the direction of $\boldsymbol{\sigma}_{1w}$) for which we know the value of the spin component; and there is none if $\boldsymbol{\sigma}_{2w}$ is not perpendicular to $\boldsymbol{\sigma}_{1w}$. If $|\boldsymbol{\sigma}_{1w}| > 1$, then we have a cone of directions for which the real part of the component of the weak vector has the value 1. (The opening angle is $\theta = 2\cos^{-1}1/|\boldsymbol{\sigma}_{1w}|$.) If also $|\boldsymbol{\sigma}_{2w}| = 0$, then these are the directions for which the spin is known; and if $|\boldsymbol{\sigma}_{2w}| \neq 0$, then the directions are the intersection of the cone and the plane, perpendicular to $\boldsymbol{\sigma}_{2w}$. We see, once again, the result which was derived in section 4 in a straightforward way: if we can ascertain the value of the spin component for more than two directions, then we can do it for a continuum of directions forming a cone.

Let us consider again the example with one particle in $N+1$ boxes discussed in section 2. We consider the same generalized state (13), but this time the number of particles in each box is measured weakly. In our situation the result of a strong measurement of the number of particles in the first N boxes is 1 with probability one. Therefore, the weak value of the number of particles in the first N boxes is also 1. But the total number of particles is just 1 (with probability one). Therefore, the weak value of the number of particles in the box $N+1$ has to be $1-N$ (it is a negative number!). Indeed, the number operator in the box $N+1$ substituted in the formula for the weak value (41) with the generalized state (13) yields $1-N$.

7. Conclusions

In quantum mechanics, in contrast to classical mechanics, even if we have complete information about the past of a certain system, future measurements add information about the system at present. The maximum possible information about the quantum

† The weak vector corresponding to the 'product'-type generalized state (3) is, however, restricted. In the appropriate basis it has the form (see equation (18) of [2]) $\boldsymbol{\sigma}_w = (1/\cos\alpha/2, 0, i\tan\alpha/2)$. Therefore, it has the following restriction: $|\boldsymbol{\sigma}_w| = 1$. We shall explain the necessity of this requirement from some general principles elsewhere.

system based on the results of experiments performed both in the past and in the future is described by the *generalized state* (18) introduced in this work. It is a complete description of a quantum system in the same sense that the quantum state in the standard approach is a complete description of a quantum system based on the results of experiments performed only in the past.

The generalized state gives probabilities for different outcomes of a measurement of any quantum variable. The probabilities are given by equation (23) (or, if the generalized state is given by equation (26), then the probabilities are given by equation (27)). The generalized state yields not only probabilities for the results of all possible conventional measurements, but also the outcomes of all possible 'weak measurements' (standard measurement procedures with 'weak enough' interaction). The weak value, the outcome of the weak measurement, is given by equation (40) (or by equation (43), in case the generalized state is given in the form (26)).

Using generalized states we were able to analyse peculiar situations in which several non-commuting variables have definite values. The existence of some of these situations has been reported before [7], but the discovery of such cases was made using the approach presented in this work. Even when one knows where to look, it is a difficult task to see these results using the standard approach [8].

We have discussed the importance of the concept of the weak value in an earlier paper [2]. However, we considered only particular situations in which complete measurements were performed on the system before and after the time in question. This caused some restrictions on the possible weak values of sets of quantum variables. The extension of the concept of the weak value to systems described by generalized states made it possible, in some situations, to make inferences about strong (conventional) measurements from the knowledge of the weak values and vice versa. The description by the generalized state yields a geometrical picture of weak values in the vector space of the complete set of quantum variables. This geometrical picture, together with the theorems which relate strong and weak measurements, provide a powerful new tool for analysing situations in which several non-commuting variables have definite values, as well as for investigating related questions.

The generalized state is the analogue of a pure quantum state for the case when the information is allowed about the future as well as about the past. One can define easily, following our formalism, the analogue of a density matrix. This corresponds to a system at time t, $t_1 < t < t_2$, such that at time t_1 a complete measurement was performed on a composite system which included our system, and at time t_2 a complete measurement was performed on *another* composite system, again including our system. This is also a complete description of a quantum system, the description of a system correlated to other systems. One can deal with this situation without defining a generalization of the density matrix by describing it as a *mixture* of generalized states. This is how one can immediately see that the theorems of section 6 are also valid in this case.

What we have presented here is a novel approach to standard quantum theory. Our formalism is in complete agreement with the standard approach in all its experimental predictions. It has an advantage that it is symmetrical under time reversal. However, we believe that our approach has more than merely aesthetic value. It simplifies the analysis of the second in a series of three consecutive experiments, i.e. the analysis of the experiment performed on ensembles which are both preselected and postselected. And, most importantly, it suggests new experiments which yield interesting results [9, 11-13]. It seems to us that it is practically impossible to anticipate these results using the standard approach.

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